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Nonsingular almost strictly sign regular matrices

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ABSTRACT

The class of nonsingular almost strictly totally positive matrices has been characterized [M. Gasca, J.M. Peña, Characterizations and decompositions of almost strictly positive matrices, SIAM J. Matrix Anal. Appl. 28 (2006) 1–8]. In this paper, we discuss the class of almost strictly sign regular matrices that includes almost strictly totally positive matrices. A characterization is provided for these matrices in terms of their nontrivial minors using consecutive rows and consecutive columns. In particular, we present a characterization of certain almost strictly sign regular matrices in terms of a very reduced number of boundary almost trivial minors.

1. Introduction

Let $\langle n\rangle = \{1, 2, \ldots, n\}$, and let $Q_{k,n}$ be the set of strictly increasing sequences of k natural numbers less than or equal to n. By a signature sequence, we mean a real sequence $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ with $|\epsilon_i| = 1$ for all $i = 1, \ldots, n$. Denote by $A[\alpha|\beta]$ the submatrix of $A \in \mathbb{R}^{n\times n}$ with rows and columns indexed by $\alpha \in Q_{k,n}$ and $\beta \in Q_{k,n}$, respectively. The principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$. For a sequence $\alpha = (\alpha_i) \in Q_{k,n}$, denote

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$$d(\alpha) = \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i - 1) = \alpha_k - \alpha_1 - (k-1)$$

with the convention $d(\alpha) = 0$ if $k = 1$.

Definition 1. Let $A \in \mathbb{R}^{n\times n}$ and $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ be a signature sequence.

(a) If $\epsilon_k \det A[\alpha|\beta] \geqslant 0$ (> 0) for all $\alpha, \beta \in Q_{k,n}$ with any $k \in \langle n\rangle$, then A is called sign regular (or strictly sign regular) with signature ϵ.
(b) If $\det A[\alpha|\beta] \geqslant 0$ (> 0) for all $\alpha, \beta \in Q_{k,n}$ with any $k \in \langle n\rangle$, then A is called totally nonnegative (or totally positive).

Sign regular matrices have a wide variety of applications in approximation theory, numerical mathematics, statistics, economics, computer aided geometric design, and others fields [16,19]. Many beautiful properties of sign regular matrices have been provided [2,5,6,15]. It has been known that it is not necessary to check the signs of all minors of a matrix to decide whether or not it is sign regular. In the last decades, reduction of the number of checked minors to determine if a given matrix is sign regular is a major topic in the study of sign regular matrices, see [1–14]. It is interesting to recall that some elegant characterizations of strictly sign regular matrices and totally positive matrices have been provided in [1,9] respectively as follows. Let $A \in \mathbb{R}^{n\times n}$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ be a signature sequence. Then

(i) A is strictly sign regular with signature ϵ if and only if for any $k \in \langle n\rangle$,

$$\epsilon_k \det A[\alpha|\beta] > 0, \ \forall\, \alpha, \beta \in Q_{k,n} \ \text{such that } d(\alpha) = d(\beta) = 0; \tag{1}$$

(ii) A is totally positive if and only if for any $k \in \langle n\rangle$,

$$\begin{cases} \det A[\alpha|1, \ldots, k] > 0, & \forall\, \alpha \in Q_{k,n} \ \text{with } d(\alpha) = 0, \\ \det A[1, \ldots, k|\beta] > 0, & \forall\, \beta \in Q_{k,n} \ \text{with } d(\beta) = 0. \end{cases} \tag{2}$$

Recently, it has been observed that the most important subclass of totally nonnegative matrices is the class of almost strictly totally positive matrices, which appears in many important applications such as approximation theory and computer aided geometric design [8]. For example, almost strictly totally positive matrices are very useful to generate bases of functions with good shape preserving properties in computer aided geometric design [19], and the collocation matrices of B-splines [2] and Hurwitz matrices [17] are almost strictly totally positive matrices. For more useful applications, the reader is referred to [8,11,12].

Definition 2 [8]**.** A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called almost strictly totally positive if and only if it is totally nonnegative and satisfies that for any $\alpha = (\alpha_i), \beta = (\beta_i) \in Q_{k,n}$ with $d(\alpha) = d(\beta) = 0$,

$$\det A[\alpha|\beta] > 0 \Leftrightarrow a_{\alpha_h,\beta_h} > 0, \quad h = 1, 2, \ldots, k.$$

Obviously, this class is intermediate between totally nonnegative matrices and totally positive matrices. In addition, Gladwell [7] introduced the class of inner totally positive matrices that is identical to that of nonsingular almost strictly totally positive matrices. To present the definition, the following notations are needed. A sequence $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is a staircase sequence if it is nondecreasing and satisfies $\gamma_i \geqslant i$ for all i. Let ρ and γ be staircase sequences. For a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, if $a_{ij} = 0$ when $i > \gamma_j$ or $j > \rho_i$, then A is called a ρ, γ-staircase matrix, and further, a minor $\det A[\alpha|\beta]$ with $\alpha = (\alpha_i), \beta = (\beta_i) \in Q_{k,n}$ is called an inner minor if $\alpha_i \leqslant \gamma_{\beta_i}, \beta_i \leqslant \rho_{\alpha_i}$ for all $i = 1, 2, \ldots, k$.

Definition 3 [7]**.** A matrix $A \in \mathbb{R}^{n\times n}$ is called inner totally positive if all the inner minors of A are positive.

The facts (1) and (2) have led to significant reduction on number of minors to be checked for determining the strictly sign regularity. Therefore, the question arises if there exist analogous results for almost strictly totally positive matrices or inner totally positive matrices. Some characterizations have been obtained in terms of a reduced number of minors [7,8,11,12]. The following result is provided by Gladwell [7], which is just a consequence of [11, Theorem 3.1] by Gasca and Peña.

Lemma 4 [7]. *Let $A \in \mathbb{R}^{n\times n}$ be a ρ, γ-staircase matrix. Then A is inner totally positive if and only if all the inner minors*

$$\det A[\alpha|\beta] > 0, \quad \forall\, k \in \langle n\rangle,\ \alpha, \beta \in Q_{k,n} \text{ such that } d(\alpha) = d(\beta) = 0.$$

Recently, Gasca and Peña [12] improved the previous characterizations in terms of a very reduced number of boundary minors that are defined as follows.

Definition 5 [12]. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, and let $B = A[\alpha|\beta]$ for $\alpha, \beta \in Q_{k,n}$ with $d(\alpha) = d(\beta) = 0$ such that $a_{\alpha_1,\beta_1}a_{\alpha_2,\beta_2}\cdots a_{\alpha_k,\beta_k} \neq 0$. Then B is a column boundary submatrix if either $\beta_1 = 1$, or $\beta_1 > 1$ and $A[\alpha|\beta_1 - 1] = 0$. Analogously, B is a row boundary submatrix if either $\alpha_1 = 1$, or $\alpha_1 > 1$ and $A[\alpha_1 - 1|\beta] = 0$. Minors corresponding to boundary submatrices are called boundary minors.

Lemma 6 [12]. *Suppose $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ satisfies that*

$$\begin{cases} a_{11} \neq 0,\ a_{22} \neq 0,\ \ldots,\ a_{nn} \neq 0, \\ a_{ij} = 0,\ i > j \ \Rightarrow\ a_{hk} = 0,\ \forall\, k \leqslant j < i \leqslant h, \\ a_{ij} = 0,\ i < j \ \Rightarrow\ a_{hk} = 0,\ \forall\, h \leqslant i < j \leqslant k. \end{cases}$$

Then A is nonsingular almost strictly totally positive if and only if all boundary minors of A are positive.

According to these results above, it is natural to pursue a characterization of a class of matrices that is intermediate between sign regular matrices and strictly sign regular matrices. Therefore, the aim of this paper is to investigate the class of almost strictly sign regular matrices. The present paper is organized as follows. In Section 2, the class of almost strictly sign regular matrices is introduced. Consequently, the class of almost strictly totally positive matrices is redefined in a simple and natural way by comparing Definitions 2 and 3. Then in Section 3, a characterization of these matrices is provided in terms of nontrivial minors using consecutive rows and consecutive columns, which, in return, implies that the class of almost strictly sign regular matrices is a proper extension of the class of strictly sign regular matrices. In Section 4, we obtain a characterization of certain almost strictly sign regular matrices with signature $\epsilon = (1, \ldots, 1, \epsilon_n)$ in terms of a very reduced number of boundary almost trivial minors. It must be remarked that our result in the case $\epsilon_n = -1$ is nontrivial, and Lemma 6 provided by Gasca and Peña corresponds to the result in the special case $\epsilon_n = 1$.

2. Definition of almost strictly sign regular matrices

In this section, we introduce the class of almost strictly sign regular matrices that includes almost strictly totally positive matrices and inner totally positive matrices. Denote by $I_n \in \mathbb{R}^{n\times n}$ the identity matrix, and let $P_n = (p_{ij}) \in \mathbb{R}^{n\times n}$ be the permutation matrix with $p_{ij} = 1$ if $i + j = n + 1$ and 0 otherwise. It must be mentioned that the following result can also be derived from [18, Lemma 2.2]. Because of its importance in this paper, we include our proof as follows.

Lemma 7. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be nonsingular sign regular with signature $\epsilon = (\epsilon_i) \in \mathbb{R}^{1\times n}$. Then the following statements hold.*

(1) If $\epsilon_2 = 1$, then

$$\begin{cases} a_{11} \neq 0,\ a_{22} \neq 0,\ \ldots,\ a_{nn} \neq 0, \\ a_{ij} = 0,\ i > j \Rightarrow a_{kl} = 0, \quad \forall\, l \leqslant j < i \leqslant k; \\ a_{ij} = 0,\ i < j \Rightarrow a_{kl} = 0, \quad \forall\, k \leqslant i < j \leqslant l; \end{cases}$$

i.e., A is of the form as follows

$$\begin{bmatrix} * & * & 0 & \cdots & 0 \\ * & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * & * \end{bmatrix},$$

which is called a type-I staircase matrix. Here, the symbol $*$ means that the corresponding entry is nonzero.

(2) If $\epsilon_2 = -1$, then

$$\begin{cases} a_{1n} \neq 0, \ a_{2,n-1} \neq 0, \ \ldots, \ a_{n1} \neq 0, \\ a_{ij} = 0, \ j > n-i+1 \ \Rightarrow a_{kl} = 0, \quad \forall\, i \leqslant k, \ j \leqslant l; \\ a_{ij} = 0, \ j < n-i+1 \ \Rightarrow a_{kl} = 0, \quad \forall\, k \leqslant i, \ l \leqslant j; \end{cases}$$

i.e., A is of the form as follows

$$\begin{bmatrix} 0 & \cdots & 0 & * & * \\ \vdots & \iddots & \iddots & \iddots & * \\ 0 & \iddots & \iddots & \iddots & 0 \\ * & \iddots & \iddots & \iddots & \vdots \\ * & * & 0 & \cdots & 0 \end{bmatrix}$$

which is called a type-II staircase matrix. Here, the symbol $*$ means that the corresponding entry is nonzero.

Proof. According to [15, Lemma 3], the case that $\epsilon_2 = 1$ is true. For the case that $\epsilon_2 = -1$, set $A' = P_nA = (a'_{ij})$. Obviously, A' is sign regular with signature $\epsilon' = (\epsilon'_i)$ where $\epsilon'_2 = 1$, and thus,

$$\begin{cases} a_{n1} = a'_{11} \neq 0, \ a_{n-1,2} = a'_{22} \neq 0, \ \ldots, \ a_{1n} = a'_{nn} \neq 0, \\ a_{ij} = a'_{n-i+1,j} = 0, \ j > n-i+1 \ \Rightarrow a_{kl} = a'_{n-k+1,l} = 0, \quad \forall\, n-k+1 \leqslant n-i+1 < j \leqslant l; \\ a_{ij} = a'_{n-i+1,j} = 0, \ j < n-i+1 \ \Rightarrow a_{kl} = a'_{n-k+1,l} = 0, \quad \forall\, l \leqslant j < n-i+1 \leqslant n-k+1. \end{cases}$$

Thus the lemma is proved. □

In the sequel, by *a trivial minor*, we mean that a minor is zero exactly according to the zero–nonzero pattern. We illustrate the fact in the following example.

Example 1. Let

$$A = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

where the symbol $*$ means that the corresponding entry is nonzero. Then $\det A[2,3,4|1,2,3]$, $\det A[2,4|1,3]$ are trivial minors, and $\det A[1,2]$, $\det A[1,3|2,4]$ are nontrivial minors.

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, and let $\alpha = (\alpha_i)$, $\beta = (\beta_i) \in Q_{k,n}$ with $k \in \langle n\rangle$. Then we have the following facts.

- If A is a type-I staircase matrix, then

$$\det A[\alpha|\beta] \text{ is a nontrivial minor} \quad \Leftrightarrow \quad a_{\alpha_1,\beta_1}a_{\alpha_2,\beta_2}\ldots a_{\alpha_k,\beta_k} \neq 0. \tag{3}$$

- If A is a type-II staircase matrix, then

$$\det A[\alpha|\beta] \text{ is a nontrivial minor} \quad \Leftrightarrow \quad a_{\alpha_1,\beta_k}a_{\alpha_2,\beta_{k-1}}\ldots a_{\alpha_k,\beta_1} \neq 0.$$

- Set $A' = P_nA$. Then A is a type-I staircase matrix if and only if A' is a type-II staircase matrix, and

$$\det A[\alpha|\beta] \text{ is a nontrivial minor} \quad \Leftrightarrow \quad \det A'[\alpha'|\beta] \text{ is a nontrivial minor}$$

  where $\alpha' = (n - \alpha_i + 1) \in Q_{k,n}$.

The definition of almost strictly sign regular matrices is presented as follows. Observe that a consequence of this is that the class of almost strictly totally positive matrices is redefined in a natural and simple way by comparing Definitions 2 and 3.

Definition 8. Let $A \in \mathbb{R}^{n\times n}$ and $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ be a signature sequence.

(i) If all the nontrivial minors

$$\epsilon_k \det A[\alpha|\beta] > 0, \quad \text{where } \alpha, \beta \in Q_{k,n} \text{ with any } k \in \langle n\rangle,$$

then A is called almost strictly sign regular with signature ϵ.

(ii) If all the nontrivial minors

$$\det A[\alpha|\beta] > 0, \quad \text{where } \alpha, \beta \in Q_{k,n} \text{ with any } k \in \langle n\rangle,$$

then A is called almost strictly totally positive.

Remark 1. Suppose $A \in \mathbb{R}^{n\times n}$ is almost strictly sign regular with signature $\epsilon = (\epsilon_i) \in \mathbb{R}^{1\times n}$. Since all trivial minors are zero, obviously A must be sign regular with signature ϵ. Moreover, when

$$\epsilon_1 \det A[\alpha|\beta] > 0, \quad \forall\, \alpha, \beta \in Q_{1,n},$$

all minors of A are nontrivial. Thus, by Definition 1, A is strictly sign regular with signature ϵ. This means that the class of almost strictly sign regular matrices is a proper extension of the class of strictly sign regular matrices. It also shows that the class of almost strictly sign regular matrices is intermediate between sign regular matrices and strictly sign regular matrices.

3. Characterization of almost strictly sign regular matrices

In this section, a characterization of nonsingular almost strictly sign regular matrices is provided in terms of their nontrivial minors using consecutive rows and consecutive columns.

Lemma 9. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a type-I staircase matrix and $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ be a signature sequence. Set $r = \min\{|j - i| : a_{ij} = 0,\ \forall\, i, j \in \langle n\rangle\} > 0$. If all the nontrivial minors*

$$\epsilon_k \det A[\alpha|\beta] > 0, \quad \forall\, k \in \langle n\rangle$$

where $\alpha, \beta \in Q_{k,n}$ such that $d(\alpha) = d(\beta) = 0$, then

$$\epsilon_2 = \epsilon_1^2, \epsilon_3 = \epsilon_1^3, \ldots, \epsilon_{n-r+1} = \epsilon_1^{n-r+1}.$$

Proof. Consider that $a_{ii} \neq 0$ for all i. Without loss of generality, assume that $a_{ts} = 0$ with $t < s$ such that $r = s - t$. Now let us prove that

$$\epsilon_k = \epsilon_1^k, \quad \forall\, 1 \leqslant k \leqslant n - r + 1$$

by induction on k. The case $k = 1$ is trivial. Assume that the result is true for k less than $n - r + 1$. Notice that A is a type-I staircase matrix. Thus, $a_{ij} = 0$ for all $i \leqslant t < s \leqslant j$. Because of the minimality of r, we have that

$$a_{1r} \neq 0, a_{2,r+1} \neq 0, \ldots, a_{n-r+1,n} \neq 0$$

which implies by (3) that $\det A[1, 2, \ldots, n-r+1|r, r+1, \ldots, n]$ is a nontrivial minor. Since

$$A[1, \ldots, n-r+1|r, \ldots, n] = \begin{bmatrix} A[1, \ldots, t|r, \ldots, s-1] & 0 \\ * & A[t+1, \ldots, n-r+1|s, \ldots, n] \end{bmatrix},$$

we have that

$$\det A[1, \ldots, n-r+1|r, \ldots, n] = \det A[1, \ldots, t|r, \ldots, s-1] \cdot \det A[t+1, \ldots, n-r+1|s, \ldots, n]$$

where each of the minors on the right is a nontrivial minor of the order less than $n - r + 1$. Applying our induction assumption, we easily get that $\epsilon_{n-r+1} = \epsilon_1^t \epsilon_1^{n-r-t+1} = \epsilon_1^{n-r+1}$. The case that $a_{ts} = 0$ with $t > s$ such that $r = t - s$ can be treated similarly. Thus the lemma is proved. $\square$

Our main result in this section is the following theorem, which is proved by using a similar argument as that of [1, Theorem 2.5].

Theorem 10. *Let $A \in \mathbb{R}^{n\times n}$ and $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ be a signature sequence. Then A is nonsingular almost strictly sign regular with signature ϵ if and only if A is a type-I or type-II staircase matrix, and all the nontrivial minors*

$$\epsilon_k \det A[\alpha|\beta] > 0, \quad \forall\, k \in \langle n\rangle \tag{4}$$

where $\alpha, \beta \in Q_{k,n}$ such that $d(\alpha) = d(\beta) = 0$.

Proof. The necessity is clear by using Lemma 7. For the sufficiency, let us prove that all the nontrivial minors

$$\epsilon_k \det A[\alpha|\beta] > 0, \ \text{ where } \alpha, \beta \in Q_{k,n} \text{ with any } k \in \langle n\rangle \tag{5}$$

by induction on k. The case $k = 1$ is trivial. Now assume that (5) is true with $k - 1$ in place of k. First fix $\alpha \in Q_{k,n}$ with $d(\alpha) = 0$, and let us prove (5) with this α by induction on $\ell = d(\beta)$. When $\ell = 0$, this follows by the assumption of the theorem. Suppose that all the nontrivial minors

$$\epsilon_k \det A[\alpha|\theta] > 0, \quad \text{where } \theta \in Q_{k,n} \text{ with } d(\theta) \leqslant \ell - 1. \tag{6}$$

Let us consider $\beta = (\beta_i) \in Q_{k,n}$ with $d(\beta) = \ell \geqslant 1$ such that $\det A[\alpha|\beta]$ is nontrivial. Set $\tau = (\beta_2, \beta_3, \ldots, \beta_{k-1})$. Since $d(\beta) = \ell \geqslant 1$, there exists $p \notin \tau$ such that $\beta_1 < p < \beta_k$. Thus, by the identity (1.39) of [1], we have

$$\begin{aligned} \det A[\omega|\tau \cup \{p\}] \det A[\alpha|\tau \cup \{\beta_1, \beta_k\}] &= \det A[\omega|\tau \cup \{\beta_k\}] \det A[\alpha|\tau \cup \{\beta_1, p\}] \\ &\quad + \det A[\omega|\tau \cup \{\beta_1\}] \det A[\alpha|\tau \cup \{p, \beta_k\}] \end{aligned} \tag{7}$$

for any $\omega \in Q_{k-1,n}$ with $\omega \subset \alpha$. If A is a type-I staircase matrix, then there are two cases.

(1) Let $a_{\alpha_t,\beta_{t+1}} \neq 0$ for all $t = 1, 2, \ldots, k-1$. Since $d(\beta) = \ell \geqslant 1$, we assume that $\beta_i < p < \beta_{i+1}$ for some $1 \leqslant i \leqslant k-1$, and so

$$d(\tau \cup \{\beta_1, p\}) \leqslant \ell - 1, \quad d(\tau \cup \{\beta_k, p\}) \leqslant \ell - 1.$$

Thus by our induction assumption (6),

$$\epsilon_k \cdot \det A[\alpha|\tau \cup \{\beta_1, p\}] \geqslant 0.$$

Notice that $\det A[\alpha|\beta]$ is nontrivial. Then by (3), $a_{\alpha_i,\beta_i} \neq 0$, which with $a_{\alpha_i,\beta_{i+1}} \neq 0$ together implies that $a_{\alpha_i,p} \neq 0$. Hence, we have

$$a_{\alpha_1,\beta_2} \neq 0, \ldots, a_{\alpha_{i-1},\beta_i}, a_{\alpha_i,p} \neq 0, a_{\alpha_{i+1},\beta_{i+1}} \neq 0, \ldots, a_{\alpha_k,\beta_k} \neq 0,$$

and thus, the minor $\det A[\alpha|\tau \cup \{p, \beta_k\}]$ is nontrivial. Using the induction assumption (6), we yield that

$$\epsilon_k \cdot \det A[\alpha|\tau \cup \{p, \beta_k\} > 0.$$

Now choose $\omega = \{\alpha_1, \ldots, \alpha_{k-1}\}$. Then

$$\begin{cases} a_{\alpha_1,\beta_2} \neq 0, \ldots, a_{\alpha_{i-1},\beta_i} \neq 0, a_{\alpha_i,p} \neq 0, a_{\alpha_{i+1},\beta_{i+1}} \neq 0, \ldots, a_{\alpha_{k-1},\beta_{k-1}} \neq 0, \\ a_{\alpha_1,\beta_1} \neq 0, \ldots, a_{\alpha_{i-1},\beta_{i-1}} \neq 0, a_{\alpha_i,\beta_i} \neq 0, a_{\alpha_{i+1},\beta_{i+1}} \neq 0, \ldots, a_{\alpha_{k-1},\beta_{k-1}} \neq 0, \end{cases}$$

from which it follows by (3) that $\det A[\omega|\tau \cup \{p\}]$ and $\det A[\omega|\tau \cup \{\beta_1\}]$ are nontrivial minors. Thus, by the assumption that (5) is true with $k-1$ in place of k,

$$\epsilon_{k-1} \cdot \det A[\omega|\tau \cup \{p\}] > 0, \quad \epsilon_{k-1} \cdot \det A[\omega|\tau \cup \{\beta_1\}] > 0, \quad \epsilon_{k-1} \cdot \det A[\omega|\tau \cup \{\beta_k\}] \geqslant 0.$$

Therefore, we obtain from (7) that $\epsilon_k \det A[\alpha|\beta] > 0$.

(2) Let $a_{\alpha_m,\beta_{m+1}} = 0$ for some $1 \leqslant m \leqslant k-1$. Since A is a type-I staircase matrix, we have that $a_{ij} = 0$ for all $i \leqslant \alpha_m < \beta_{m+1} \leqslant j$, and thus

$$A[\alpha|\beta] = \begin{bmatrix} A[\alpha_1, \ldots, \alpha_m|\beta_1, \ldots, \beta_m] & 0 \\ * & A[\alpha_{m+1}, \ldots, \alpha_k|\beta_{m+1}, \ldots, \beta_k] \end{bmatrix}.$$

Hence, the nontrivial minor

$$\det A[\alpha|\beta] = \det A[\alpha_1, \ldots, \alpha_m|\beta_1, \ldots, \beta_m] \cdot \det A[\alpha_{m+1}, \ldots, \alpha_k|\beta_{m+1}, \ldots, \beta_k],$$

where each of the minors on the right is a nontrivial minor of the order less than k. According to Lemma 9, since $a_{\alpha_m,\beta_{m+1}} = 0$, the signature sequence ϵ satisfies that

$$\epsilon_2 = \epsilon_1^2, \epsilon_3 = \epsilon_1^3, \ldots, \epsilon_{n-\beta_{m+1}+\alpha_m+1} = \epsilon_1^{n-\beta_{m+1}+\alpha_m+1}.$$

Notice that $k \leqslant \alpha_m + (n - \beta_{m+1} + 1)$. Thus, by the assumption that (5) is true with $k-1$ in place of k, we get that

$$\begin{aligned} \epsilon_k \det A[\alpha|\beta] = \epsilon_1^k \det A[\alpha|\beta] = \epsilon_1^m & \det A[\alpha_1, \ldots, \alpha_m|\beta_1, \ldots, \beta_m] \\ & \times \epsilon_1^{k-m} \det A[\alpha_{m+1}, \ldots, \alpha_k|\beta_{m+1}, \ldots, \beta_k] > 0. \end{aligned}$$

Therefore, we prove that (5) is true when $\alpha \in Q_{k,n}$ with $d(\alpha) = 0$. Apply the same argument rowwise to conclude that (5) is generally true. Thus, A is almost strictly sign regular with the desired signature ϵ.

For the case that A is a type-II staircase matrix, let $A' = P_n A$, then A' is a type-I staircase matrix. Because of (4), all the nontrivial minors of A'

$$(-1)^{\frac{k(k-1)}{2}}\epsilon_k \mathrm{det}A'[\alpha'|\beta] > 0, \quad \forall\, k \in \langle n\rangle$$

where $\alpha' = (n - \alpha_i + 1)$, $\beta = (\beta_i) \in Q_{k,n}$ with $d(\alpha') = d(\beta) = 0$. Therefore, according to the case above, we have that A' is almost strictly sign regular with signature $\epsilon' = ((-1)^{\frac{i(i-1)}{2}}\epsilon_i) \in \mathbb{R}^{1\times n}$. Thus, A is almost strictly sign regular with the desired signature ϵ. $\square$

Obviously, Lemma 4 and [11, Theorem 3.1] correspond to our Theorem 10 in the special case $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_n = 1$.

Remark 2. If the condition (4) is satisfied with $\epsilon_1 \mathrm{det}A[\alpha|\beta] > 0$ for all $\alpha, \beta \in Q_{1,n}$, then all minors of A are nontrivial, and thus,

$$\epsilon_k \mathrm{det}A[\alpha|\beta] > 0,\ \forall\, k \in \langle n\rangle,\ \ \alpha, \beta \in Q_{k,n}\ \text{ such that } d(\alpha) = d(\beta) = 0.$$

By Theorem 10, A is almost strictly sign regular with signature ϵ. However, the fact (1) implies that A is strictly sign regular with signature ϵ. Hence, our Theorem 10 is the natural extension of the fact (1) on strictly sign regular matrices. Thus, our result again shows that the class of almost strictly sign regular matrices is a proper extension of the class of strictly sign regular matrices.

4. Characterization of certain almost strictly sign regular matrices

In this section, we provide a characterization of nonsingular almost strictly sign regular matrices with signature $\epsilon = (1, \ldots, 1, \epsilon_n)$ $(n \geqslant 3)$ in terms of the signs of boundary almost trivial minors that are defined by referring to [12] as follows.

Definition 11. Let $A \in \mathbb{R}^{n\times n}$ and $\alpha = (\alpha_i), \beta = (\beta_i) \in Q_{k,n}$. Then $\mathrm{det}A[\alpha|\beta]$ is called a boundary almost trivial minor ($A[\alpha|\beta]$ is called a boundary almost trivial submatrix) if the following conditions are satisfied:

(i) $\mathrm{det}A[\alpha|\beta]$ is a nontrivial minor with $d(\alpha) = d(\beta) = 0$;
(ii) $\mathrm{det}A[\{\alpha_1 - 1\} \cup \alpha|\{\beta_1 - 1\} \cup \beta]$ is a trivial minor; otherwise $\alpha_1 = 1$ or $\beta_1 = 1$.

In particular, if $\alpha_k > \beta_k$, then $\mathrm{det}A[\alpha|\beta]$ is called a lower boundary almost trivial minor; if $\alpha_k < \beta_k$, then $\mathrm{det}A[\alpha|\beta]$ is called an upper boundary almost trivial minor.

The following example is used to illustrate a slight difference between boundary almost trivial minors and boundary minors of Gasca and Peña.

Example 2. Let

$$A = \begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix},$$

where the symbol $*$ means that the corresponding entry is nonzero. Then all the boundary almost trivial minors of A are the following

$$a_{11}, a_{12}, a_{21}, a_{34},\ \mathrm{det}A[1, 2],\ \mathrm{det}A[2, 3|1, 2],\ \mathrm{det}A[1, 2, 3],\ \mathrm{det}A[2, 3, 4|1, 2, 3],\ \mathrm{det}A.$$

Beside all the minors above, all the boundary minors of A also include the following minors:

$$a_{32}, a_{33}, a_{43},\ \mathrm{det}A[3, 4|2, 3],\ \mathrm{det}A[3, 4].$$

For any $\alpha = (\alpha_i), \beta = (\beta_i) \in Q_{k,n}$, if $\alpha_i \leqslant \beta_i$ for all i, then we denote $\alpha \leqslant \beta$; and further $\alpha < \beta$ if $\alpha \neq \beta$. To present our main result in this section, we need the following lemmas.

Lemma 12 [1]. *If $A \in \mathbb{R}^{n\times n}$ is a nonsingular totally nonnegative matrix, then for any $k \in \langle n\rangle$,*

$$\det A[\alpha] > 0, \quad \forall\, \alpha \in Q_{k,n}.$$

Lemma 13 [14]. *Let $A \in \mathbb{R}^{n\times n}$ be a nonnegative tridiagonal matrix. If*

$$\det A \geqslant 0,$$

$$\det A[1, \ldots, n-1] \geqslant 0,$$

$$\det A[1, \ldots, k] > 0, \quad \forall\, k \in < n-2 >,$$

then A is totally nonnegative.

Lemma 14. *Let $A \in \mathbb{R}^{n\times n}$ be a type-I staircase matrix. If all boundary almost trivial minors of orders less than n are positive, then A can be factorized as*

$$A = LTU = \begin{bmatrix} 1 & \\ & \tilde{L} \end{bmatrix} T \begin{bmatrix} 1 & \\ & \tilde{U} \end{bmatrix},$$

where L is unit lower triangular totally nonnegative, U is unit upper triangular totally nonnegative, and T is tridiagonal. Furthermore, if $\det A > 0$ *or* $\det A[2, 3, \ldots, n] > 0$, *then T is nonnegative.*

Proof. Let $E_j(x)$ be an elementary matrix obtained from an identity matrix I_n by changing the $(j, j-1)$th entry to x. Since A is a type-I staircase matrix, there exist integers $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_{n-1}$ and $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_{n-1}$ such that

$$\begin{aligned} &r_j \geqslant j,\ a_{r_j,j} \neq 0,\ a_{r_j+1,j} = \ldots = a_{nj} = 0,\ j = 1, 2, \ldots, n-1; \\ &s_i \geqslant i,\ a_{i,s_i} \neq 0,\ a_{i,s_i+1} = \ldots = a_{in} = 0,\ i = 1, 2, \ldots, n-1. \end{aligned} \tag{8}$$

Set $B_0 = A = (b_{ij}^{(0)})$. Consider the condition that all boundary almost trivial minors of orders less than n are positive. Thus,

$$b_{11}^{(0)} = a_{11} > 0, \ldots, b_{r_1,1}^{(0)} = a_{r_1,1} > 0\ (r_1 \geqslant 2); \quad b_{21}^{(0)} = b_{31}^{(0)} = \cdots = b_{n1}^{(0)} = 0\ (r_1 = 1), \tag{9}$$

and so we have a unit lower triangular totally nonnegative matrix

$$L_1 = E_{r_1}\left(\frac{b_{r_1,1}^{(0)}}{b_{r_1-1,1}^{(0)}}\right) \cdots E_3\left(\frac{b_{31}^{(0)}}{b_{21}^{(0)}}\right) = \operatorname{diag}(1, \tilde{L}_1), \quad \text{or } L_1 = I_n$$

such that $B_1 = L_1^{-1}B_0 = (b_{ij}^{(1)})$ satisfies that $b_{i1}^{(1)} = 0$ for all $i > 2$. Since the lower boundary almost trivial minors satisfy

$$\det A[2, 3|1, 2] > 0, \quad \det A[3, 4|1, 2] > 0, \ldots, \det A[r_1 - 1, r_1|1, 2] > 0,$$

$$a_{r_1+1,2} > 0, \ldots, a_{r_2,2} > 0,$$

we have

$$\begin{cases} b_{i2}^{(1)} = \frac{\det A[i-1,i|1,2]}{a_{i-1,1}} > 0,\ \forall\, 3 \leqslant i \leqslant r_1, \\ b_{i2}^{(1)} = a_{i2} > 0,\ \forall\, r_1 < i \leqslant r_2, \\ b_{i2}^{(1)} = a_{i2} = 0,\ \forall\, r_2 < i \leqslant n, \end{cases} \quad (r_2 \geqslant 3); \quad b_{32}^{(1)} = b_{42}^{(1)} = \cdots = b_{n2}^{(1)} = 0\ (r_2 = 2). \tag{10}$$

Hence, we have a unit lower triangular totally nonnegative matrix

$$L_2 = E_{r_2}\left(\frac{b^{(1)}_{r_2,2}}{b^{(1)}_{r_2-1,2}}\right)\cdots E_4\left(\frac{b^{(1)}_{42}}{b^{(1)}_{32}}\right) = \mathrm{diag}(1, 1, \tilde{L}_2), \quad \text{or } L_2 = I_n$$

such that $B_2 = L_2^{-1}B_1 = (b^{(2)}_{ij})$ satisfies that $b^{(2)}_{ij} = 0$ for all $j = 1, 2$ and $i = j+2, \dots, n$. More generally, by repeating these procedures, for $1 \leqslant l \leqslant n-3$, we can get $B_l = (b^{(l)}_{ij})$ satisfies that

$$\begin{cases} b^{(l)}_{i,l+1} = \frac{\det A[i-l+m-1,\dots,i-1,i|m,\dots,l,l+1]}{\det A[i-l+m-1,\dots,i-1|m,\dots,l]} > 0, \ \forall\, l+2 \leqslant i \leqslant r_l, \\ b^{(l)}_{i,l+1} = a_{i,l+1} > 0, \ \forall\, r_l < i \leqslant r_{l+1}, \\ b^{(l)}_{i,l+1} = a_{i,l+1} = 0, \ \forall\, r_{l+1} < i \leqslant n, \end{cases} \qquad (r_{l+1} \geqslant l+2), \tag{11}$$

where the numerator and denominator are lower boundary almost trivial minors of orders less than n; or

$$b^{(l)}_{l+2,l+1} = b^{(l)}_{l+3,l+1} = \cdots = b^{(l)}_{n,l+1} = 0 \quad (r_{l+1} = l+1), \tag{12}$$

and thus we have a unit lower triangular totally nonnegative matrix

$$L_{l+1} = E_{r_{l+1}}\left(\frac{b^{(l)}_{r_{l+1},l+1}}{b^{(l)}_{r_{l+1}-1,l+1}}\right)\dots E_{l+3}\left(\frac{b^{(l)}_{l+3,l+1}}{b^{(l)}_{l+2,l+1}}\right) = \mathrm{diag}(\underbrace{1, 1, \dots, 1}_{l+1}, \tilde{L}_{l+1}), \quad \text{or} \quad L_{l+1} = I_n$$

such that $B_{l+1} = L_{l+1}^{-1}B_l = (b^{(l+1)}_{ij})$ satisfies that

$$b^{(l+1)}_{ij} = 0, \quad \forall\, j = 1, \dots, l+1, \ i = j+2, \dots, n.$$

Therefore, we conclude that there exists a unit lower triangular totally nonnegative matrix

$$L = L_1 L_2 \dots L_{n-2} = \mathrm{diag}(1, \tilde{L})$$

such that $B = L^{-1}A = (b_{ij})$ is an upper Hessenberg matrix. In particular, it is not difficult to see by considering (8) that for all $i = 1, 2, \dots, n-1$,

$$b_{i+1,s_i+1} = a_{i+1,s_i+1}, \dots, b_{i+1,n} = a_{i+1,n}. \tag{13}$$

Next we focus on the upper Hessenberg matrix $B = T_0 = (t^{(0)}_{ij})$. Since

$$t^{(0)}_{11} = a_{11} > 0, \dots, t^{(0)}_{1,s_1} = a_{1,s_1} > 0, \ t^{(0)}_{1,s_1+1} = \dots = t^{(0)}_{1n} = 0 \quad (s_1 > 1);$$

$$t^{(0)}_{12} = \dots = t^{(0)}_{1n} = 0 \quad (s_1 = 1), \tag{14}$$

we have a unit lower triangular totally nonnegative matrix

$$U_1 = E_3^T\left(\frac{t^{(0)}_{13}}{t^{(0)}_{12}}\right)\cdots E_{s_1}^T\left(\frac{t^{(0)}_{s_1,1}}{t^{(0)}_{s_1-1,1}}\right) = \mathrm{diag}(1, \tilde{U}_1), \quad \text{or} \quad U_1 = I_n$$

such that $T_1 = BU_1^{-1} = (t^{(1)}_{ij})$ satisfies that

$$t^{(1)}_{1j} = 0, \ \text{for all } j > 2; \quad t^{(1)}_{ij} = 0, \ \forall\, i-j > 1.$$

Now given any upper boundary almost trivial minor $\det A[\alpha|\beta]$, where $\alpha = (\alpha_i) = (s, s+1, \dots, k+1)$ and $\beta = (\beta_i) = (j-k+s-1, \dots, j-1, j)$ with $j > k+1$. We consider the following two cases.

- Assume $s = 1$. Then it is easy to check by the Cauchy–Binet formula [1] that

$$\det B[\alpha|\beta] = \det(L^{-1}A)[\alpha|\beta] = \det A[\alpha|\beta] > 0.$$

- Assume $s > 1$. Then $\det A[\{\alpha_1 - 1\} \cup \alpha|\{\beta_1 - 1\} \cup \beta]$ is a trivial minor. By (3), $a_{s-1,j-k+s-2} = 0$. Notice that $j > k + 1$, this means that $j - k + s - 2 > s - 1$. Thus, $a_{ih} = 0$ for all $i \leqslant s - 1$ and $h \geqslant j - k + s - 2$ since A is a type-I staircase matrix, and so

$$A[1, 2, \ldots, s-1|j-k+s-2, \ldots, n-1, n] = 0,$$

from which it is easy to see that

$$B[1, 2, \ldots, s-1|j-k+s-2, \ldots, n-1, n] = 0. \tag{15}$$

Thus, $\det B[\{\alpha_1 - 1\} \cup \alpha|\{\beta_1 - 1\} \cup \beta] = 0$. Moreover, by the Cauchy–Binet formula,

$$\begin{aligned}
\det B[\alpha|\beta] &= \sum_{\omega \in Q_{k-s+2,n}} \det L^{-1}[\alpha|\omega] \det A[\omega|\beta] \\
&= \sum_{\omega \in Q_{k-s+2,n}, \omega \leqslant \alpha} \det L^{-1}[\alpha|\omega] \det A[\omega|\beta] \\
&= \det L^{-1}[\alpha|\alpha] \det A[\alpha|\beta] + \sum_{\omega \in Q_{k-s+2,n}, \omega < \alpha} \det L^{-1}[\alpha|\omega] \det A[\omega|\beta] \\
&= \det A[\alpha|\beta] > 0
\end{aligned}$$

because $\det A[\omega|\beta] = 0$ for any $\omega < \alpha$. Similarly, we have

$$\det B[s, s+1, \ldots, k|j-k+s-1, \ldots, j-1] = \det A[s, s+1, \ldots, k|j-k+s-1, \ldots, j-1] > 0.$$

This means that if $\det A[\alpha|\beta] > 0$ is an upper boundary almost trivial minor, then $\det B[\alpha|\beta] > 0$ and $\det B[\{\alpha_1 - 1\} \cup \alpha|\{\beta_1 - 1\} \cup \beta] = 0$ for $\alpha_1 > 1$. Hence, when using the similar procedures above to eliminate the entry in the position (i, j) with $j - i > 1$ of B, we get by considering (15) and (13) that $T_k = (t_{ij}^{(k)})$ $(1 \leqslant k \leqslant n-3)$ satisfies the following

$$\begin{cases}
t_{k+1,j}^{(k)} = \frac{\det B[s,s+1,\ldots,k,k+1|j-k+s-1,\ldots,j-1,j]}{\det B[s,s+1,\ldots,k|j-k+s-1,\ldots,j-1]} \\
\qquad = \frac{\det A[s,s+1,\ldots,k,k+1|j-k+s-1,\ldots,j-1,j]}{\det A[s,s+1,\ldots,k|j-k+s-1,\ldots,j-1]} > 0, \ \forall\, k+2 \leqslant j \leqslant s_k, \\
t_{k+1,j}^{(k)} = b_{k+1,j} = a_{k+1,j} > 0, \ \forall\, s_k < j \leqslant s_{k+1}, \\
t_{k+1,j}^{(k)} = b_{k+1,j} = a_{k+1,j} = 0, \ \forall\, s_{k+1} < j \leqslant n,
\end{cases} \quad (s_{k+1} \geqslant k+2) \tag{16}$$

or

$$t_{k+1,k+2}^{(k)} = t_{k+1,k+3}^{(k)} = \ldots = t_{k+1,n}^{(k)} = 0 \quad (s_{k+1} = k+1), \tag{17}$$

and thus we have a unit lower triangular totally nonnegative matrix

$$U_{k+1} = E_{k+3}^T \left(\frac{t_{k+1,k+3}^{(k)}}{t_{k+1,k+2}^{(k)}} \right) \cdots E_{s_{k+1}}^T \left(\frac{t_{k+1,s_{k+1}}^{(k)}}{t_{k+1,s_{k+1}-1}^{(k)}} \right) = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{k+1}, \tilde{U}_{k+1}), \quad \text{or} \quad U_{k+1} = I_n$$

such that $T_{k+1} = T_k U_{k+1}^{-1} = (t_{ij}^{(k+1)})$ satisfies that

$$t_{ij}^{(k+1)} = 0, \ \forall\, i = 1, \ldots, k+1, \ j = i+2, \ldots, n; \quad t_{ij}^{(k+1)} = 0, \ \forall\, i - j > 1.$$

Therefore, there exists an upper triangular totally nonnegative matrix

$$U = U_{n-2} \ldots U_2 U_1 = \mathrm{diag}(1, \tilde{U})$$

such that $T = BU^{-1} = (t_{ij})$ is tridiagonal. Hence $A = LTU$.

Next we show that if $\det A > 0$ or $\det A[2, 3, \ldots, n] > 0$, then T is nonnegative. From the argument above, it is easy to see that

$$t_{i+1,i} = b_{i+1,i}^{(i-1)}, \quad t_{i,i+1} = t_{i,i+1}^{(i-1)}, \quad \forall\, i = 1, 2, \ldots, n-1,$$

and thus by considering (9)–(17), we have that $t_{ij} \geqslant 0$ for all $|i-j| = 1$. By the Cauchy–Binet formula, the boundary almost trivial minors

$$\det A[1, \ldots, k] = \det T[1, \ldots, k] > 0, \quad \forall\, k \in < n-1 > .$$

Obviously, the properties that $t_{12} \geqslant 0$, $t_{21} \geqslant 0$, $\det T[1] = t_{11} > 0$ and $\det T[1, 2] = t_{11}t_{22} - t_{12}t_{21} > 0$ yield that $t_{22} > 0$. Further,

$$\begin{cases} t_{k,k-1} \geqslant 0, \quad t_{k-1,k} \geqslant 0, \\ t_{kk}\det T[1, \ldots, k-1] - t_{k,k-1}t_{k-1,k}\det T[1, \ldots, k-2] \quad \forall\, k \in < n-1 >, \\ \quad = \det T[1, \ldots, k] > 0, \end{cases}$$

from which we obtain that

$$t_{kk} > 0, \quad k = 3, \ldots, n-1.$$

- If $\det A > 0$, then $\det T = \det A > 0$, and so it follows from the fact

  $$\begin{cases} t_{n,n-1} \geqslant 0, \quad t_{n-1,n} \geqslant 0, \\ t_{nn}\det T[1, \ldots, n-1] - t_{n,n-1}t_{n-1,n}\det T[1, \ldots, n-2] = \det T > 0, \end{cases}$$

  that $t_{nn} > 0$.
- If $\det A[2, 3, \ldots, n] > 0$, then

  $$0 < \det A[2, 3, \ldots, n] = \det T[2, 3, \ldots, n].$$

  By Lemma 13, $T[1, 2, \ldots, n-1]$ is nonsingular totally nonnegative, and thus, the properties

  $$\begin{cases} t_{n,n-1} \geqslant 0, \quad t_{n-1,n} \geqslant 0, \\ t_{nn}\det T[2, \ldots, n-1] - t_{n,n-1}t_{n-1,n}\det T[2, \ldots, n-2] = \det T[2, \ldots, n] > 0 \end{cases}$$

  yield $t_{nn} > 0$.

So T is nonnegative and tridiagonal. Thus, the lemma is proved. $\square$

Lemma 15 [14]. *Let $T \in \mathbb{R}^{n\times n}$ $(n \geqslant 3)$ be a nonnegative tridiagonal matrix and $\epsilon = (1, \ldots, 1, \epsilon_n)$ be a signature sequence. Then T is sign regular with signature ϵ if and only if the following conditions are satisfied:*

$$\begin{cases} \epsilon_n \det T > 0, \\ \det T[2, \ldots, n] \geqslant 0, \\ \det T[1, \ldots, n-1] \geqslant 0, \\ \det T[1, \ldots, k] > 0, \quad \forall\, 1 \leqslant k \leqslant n-2. \end{cases}$$

Theorem 16. *Let $A \in \mathbb{R}^{n\times n}$ $(n \geqslant 3)$ and $\epsilon = (1, \ldots, 1, \epsilon_n)$ be a signature sequence. Then A is nonsingular almost strictly sign regular with signature ϵ if and only if A is a type-I staircase matrix, all boundary almost trivial minors of orders less than n are positive, and*

$$\epsilon_n \det A > 0, \quad \det A[2, 3, \ldots, n] > 0.$$

Proof. Obviously, the necessity is true by considering Lemma 7. Conversely, according to Lemma 14, A can be factorized as

$$A = LTU = \begin{bmatrix} 1 & \\ & \tilde{L} \end{bmatrix} T \begin{bmatrix} 1 & \\ & \tilde{U} \end{bmatrix},$$

where L is unit lower triangular totally nonnegative, U is unit upper triangular totally nonnegative, and $T = (t_{ij})$ is tridiagonal and nonnegative. Thus we easily have, by Lemma 15, that T, and hence A, is sign regular with signature $\epsilon = (1, \ldots, 1, \epsilon_n)$. To prove the sufficiency, by Theorem 10, it suffices to show that for all $k \in \langle n-1 \rangle$, any nontrivial minor

$$\det A[\alpha|\beta] > 0, \quad \text{where } \alpha, \beta \in Q_{k,n} \text{ such that } d(\alpha) = d(\beta) = 0.$$

Without loss of generality, assume that $\alpha = (i, i+1, \ldots, i+k-1)$ and $\beta = (j, j+1, \ldots, j+k-1)$. There are the following two cases that we need to consider:

- Suppose $i \neq j$. If $i > j$, then there exists $1 \leqslant m \leqslant j$ such that $A[\alpha|\beta]$ is the principal submatrix of the boundary almost trivial submatrix

$$B = A[i-j+m, i-j+m+1, \ldots, i+k-1 | m, m+1, \ldots, j+k-1].$$

Since $i > j$, the order of B is at most $n-1$. So $\det B > 0$. Thus, since A is sign regular with signature $\epsilon = (1, \ldots, 1, \epsilon_n)$, we have that B is nonsingular totally nonnegative. Then it holds using Lemma 12 that $\det A[\alpha|\beta] > 0$. Similarly, we have $\det A[\alpha|\beta] > 0$ if $i < j$.
- Suppose $i = j$. If $i + k - 1 \leqslant n - 1$, then we easily get that $A[\alpha|\beta]$ is a principal submatrix of the boundary almost trivial submatrix $A[1, 2, \ldots, n-1]$ that is nonsingular totally nonnegative; if $i + k - 1 = n$, then $A[\alpha|\beta]$ is a principal submatrix of the submatrix $A[2, 3, \ldots, n]$ that is nonsingular totally nonnegative. Thus, by using Lemma 12, we have that $\det A[\alpha|\beta] > 0$.

Therefore, we conclude by Theorem 10 that A is almost strictly sign regular with signature $\epsilon = (1, \ldots, 1, \epsilon_n)$. $\square$

Remark 3. For the sufficiency of Theorem 16 in the case $\epsilon_n = -1$, the condition that $\det A[2, 3, \ldots, n] > 0$ is essential. We illustrate this fact by the following example. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 7 \\ 1 & 6 & 7 \end{bmatrix}.$$

Obviously, $\det A < 0$, and A is a type-I staircase matrix with all boundary almost trivial minors of orders less than 3 being positive. But, $\det A[2, 3] = -14$. So A is not sign regular.

For the sufficiency of Theorem 16 in the case $\epsilon_n = 1$, the condition that $\det A[2, 3, \ldots, n] > 0$ can be removed because the remaining conditions can ensure by using Lemma 14 and Lemma 13 that the tridiagonal matrix T, and hence A, is totally nonnegative. Observe from Definitions 5 and 11 that if $\det A[\alpha|\beta]$ is a boundary almost trivial minor, then $\det A[\alpha|\beta]$ is either a boundary minor, or a product of boundary minors. Therefore, we immediately have the following result, which is very close to Lemma 6 by Gasca and Peña.

Corollary 17 [12]. *Let $A \in \mathbb{R}^{n \times n}$. Then A is nonsingular almost strictly totally positive if and only if A is a type-I staircase matrix, and all boundary almost trivial minors are positive.*

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